Markov Processes

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Imagine a house with three rooms, A, B and C. People can wander around the three rooms, but it has been checked that, after one minute, 20% of the people in room A will go to room B, 30% of the people will move to room C, and the rest will remain there. Similar numbers hold for the rest of the rooms, so we can display the numbers in a table:

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \end{array}$$

The table must be read with the row as the "from" and he column as the "to". So, the fraction of people going from room C to room A after one minute is 0.4. Notice that all columns add up to one. This is in agreement with Geneva convention that no people should be killed in mathematical processes.

We know that there are 1000 people in room A. After some minutes, where will those people be? 20% will have moved to B and 30% to C, so we get $A = 500$, $B = 200$ and $C = 300$. Now we wait for another minute. Then, the people from A will have spread further: 250 will remain, 100 will move to B and 150 to C. But people from B will also move! They were 200, and 60% will remain, so 120 people will stay. 10% of the people at B will move to A, so we have 10% of 200, 20 people. And 30% of them will move to C, so 60 people will go. And now, C... this is boring, but let's finish it out. 40% of the people at C will go to A, so 40% of 300 is 120. 20% will go to B, so 60. And the remaining 40% will stay, so 120 people... now we have to sum up!!

New people at A: people remaining at A + people coming from B + people coming from C: $250 + 20 + 120 = 390$.

New people at B: people remaining at B + people coming from A + people coming from C: $120 + 100 + 120 = 340$.

New people at C: people remaining at C + people coming from A + people coming from B: $120 + 150 + 60 = 330$.

Uff... This way is hard to follow. Let's invent a better one. We can put all three values in a vector. This way, the starting distribution of people would be given by $P(0) = (1000, 0, 0)$. After one minute we have $P(1) = (500, 200, 300)$ and after another minute we get $P(2) = (390, 340, 330)$.

How to obtain a vector from the previous one? Easy: just multiply the matrix we wrote above by the vector. Let's try it out:

$$\begin{pmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 500 \\ 200 \\ 300 \end{pmatrix}$$

In general terms, if we call M to that matrix, we have

$$P(t+1) = M \cdot P(t)$$

The interpretation in terms of population has a problem. After a few more time steps, the numbers get fractional. We should interpret $P(t)$ as a **probability vector**, a vector with all positive entries, normalized so that the sum of the values is one.

E1. Find $P(3)$, normalized as a probability vector.

E2. Proof that, if $P(t)$ is a probability vector, $P(t+1)$ is also. Which property of M did you need?

Now suppose that we're asked about the long term behaviour of the system. For very very long times, we may assume that some kind of equilibrium has been reached. This does not mean that people are not moving, of course they will move, but the amount of people at each room will stay more or less the same. In that moment, $P(t+1)$ and $P(t)$ coincide, so we want to find the vector P_{eq} such that

$$MP_{eq} = P_{eq}$$

So we can say that M, in order to reach equilibrium, must have an eigenvalue 1. Is this true in general? If we substract the identity we get another matrix

$$(M - I)P_{eq} = 0$$

So $M - I$ must have nontrivial kernel (or an eigenvalue zero). Let's solve that equation in our case. We get that the system is compatible and indeterminate, with solution (surprisingly enough) $R_1 = R_2 = R_3$. Therefore, in equilibrium, the system has equal probabilities. Normalizing, $R_1 = R_2 = R_3 = 1/3$.

By the way, stochastic processes like this are called **Markov processes**. The probability for the next time-step depends uniquely on the probabilities at this time: the system has no memory of the past. Also, matrices like M are called **stochastic matrices**. The defining properties are that all columns should add up to one and all entries should be positive. Then, they can be the associated matrix of a Markov process.

E3. Let's try another system... Imagine again three states, with transition probabilities $P_{1\to 2} = 0.4$, $P_{1\to 3} = 0.5$, $P_{2\to 1} = P_{2\to 3} = P_{3\to 2} = 0.1$. Guess intuitively the equilibrium state and find it out exactly.

Imagine that we want to find the complete time evolution for the probability vector. Then we start by noting that

$$P(2) = M \cdot P(1) = M \cdot M \cdot P(0) = M^2 P(0)$$

And, therefore,

$$P(n) = M^n P(0)$$

So, if we can raise the matrix to te appropriate power, we can find exactly the time evolution of the probability vector! This can be easily done if the matrix is diagonalizable. Then,

$$M = SDS^{-1}$$

where D is diagonal (with, possibly, complex entries) and S *need not* be unitary. Then,

$$M^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1}$$

I.e.: we can compute any power of M by just raising the eigenvalues to that power. So, let us diagonalize the stochastic matrix. Remember an advantage that we have: we *know* that the eigenvalue 1 is there!

E4. Compute the eigenvalues and eigenvectors of the stochastic matrix. As a check: they are 1, 0.1 and 0.4.

E5. Compute the matrices M^2 and M^{10}, and compute $P(2)$ and $P(10)$ for the given initial vector.

E6. Study again the long term behaviour of the system.

E7. Every year, $2/10$ of the population in Madrid move to the rest of Spain, and $1/10$ of the population of Spain move to Madrid. Find the equilibrium state and the evolution for all time.