Stokes Theorem

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Introduction

Line, surface and volume integrals are related through a very beautiful theorem with three forms. It is globally called Stokes theorem, and is used extensively in many branches of physics: electromagnetism, fluid mechanics, gravitation... It is also one of the most beautiful and elegant theorems in mathematics, if correctly understood. In this text we will try to give the practical working tool along with an introduction to its depth and beauty...

Gradient and Curl

Let $v(x,y,z)$ be a vector field in $\mathbf{R}^3$. We're asked the following question: how to know whether it is the gradient of some scalar function? I mean: is there a certain scalar $f(x,y,z)$, such that $v = \nabla f$?

Not every v is a gradient! To see this, remember that the crossed derivatives should coincide:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Therefore, the mixed derivatives of v should coincide:

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}$$

We will adopt a more convenient notation: $A_{,x}$ means the partial derivative with respect to x of any object. So,

$$v_{x,y} - v_{y,x} = 0$$

This should be true for all couples of coordinates: xy, yz and zx. In principle, in order to summarize, we can write all these three things as a vector:

$$(v_{x,y} - v_{y,x}, v_{y,z} - v_{z,y}, v_{z,x} - v_{x,z})$$

If not all three components of this vector are zero, then the vector function v can't be the gradient of any function. That's clear. This vector is going to be important, so we will give a name to it. We will call it the **curl** of v, and denoted normally as $\nabla \times v$. Why? Because it can be written in shorthand like

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix}$$

An example is in order... Consider the vector field:

$$v(x,y,z) = (yz, xz, xy)$$

May this vector field be the gradient of a certain scalar function? Let's compute its curl:

$$\nabla \times v = (z - z, x - x, y - y) = (0,0,0)$$

OK, so it is the gradient of some function. How to obtain it? Let us call that function $f(x,y,z)$, such that $\nabla f = v$. Then, we know that the partial derivative of that thing with respect to x is yz. This means that the function is xyz plus any function in which x does not appear. Let us call that thing $xyz + C_1(y,z)$. Now, the partial derivative with respect to y is xz, so the function is $xyz + C_2(x,z)$ for the same reason. A little bit of thought gives us that the function is simply $f(x,y,z) = xyz$.

(The converse is also true: under very general conditions, if the curl of a vector field is zero, then it is the gradient of some function. But we won't get into proving that now...)

Line Integral of a Gradient

If a vector field is the gradient of a scalar function, then line integrals are very easy to carry out. Let's see why. Remember that, if you start at a certain point r_0 and make a small motion dr, then the change in a (scalar) function is just $df = \nabla f \cdot dr$. So, you can say that

$$\int_C \nabla f \cdot dr = \int_C df$$

you're adding up the small increments in f along the function! So, if path C goes from point r_0 to r_1, the total integral is

$$\int_C \nabla f \cdot dr = f(r_1) - f(r_0)$$

So the line integral depends only on the initial and final points, not on the exact path! Another conclusion: if you take the line integral of a gradient along a *closed* contour, then you get a beautiful zero. We can mix up both results and say:

- If $\nabla \times v = 0$, then the line integral of v along any closed contour is zero.

That's nice. Now let us remember the physical interpretation of the line integral of a force field as the work carried out by the force when a particle moves on a certain trajectory. If the force is the gradient of a certain potential energy function (we say $f = -\nabla V$), then the work only depends on the initial and final points. We say that the force field is *conservative*. If the particle takes a full loop, the total work does not change. Remember that that work goes into kinetic energy of the particle, so we conclude that, after a loop, the kinetic energy of the particle is preserved.

E1. Prove that any central force is conservative, i.e.: if the force only depends on r and is always pointing in the radial line.

But... what is the Curl?

First of all, let us remark one thing: the curl of a vector field can be written as a vector again only in 3D! We have three conditions because we have three pairs of variables. In 2D we have only one condition. And, in 4D, we would have six of them.

Once said that, let us consider a very very small closed circuit Γ in the xy plane. Starting from point (x, y), it advances dx in the x-direction, then it advances dy in the y-direction. Then it goes back $-dx$, and then back $-dy$. Thus, it returns to the original point. Look at the picture:

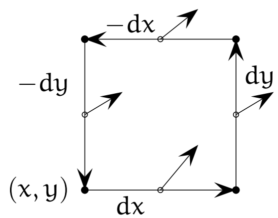

Figure 1. Line integral on a very small closed circuit.

Now we'll attempt to find the line integral of a vector field v along that circuit. We will assume that the square is so small that, in each edge, the vector field is constant, and equal to the value of the field at the middle point. So, as we proceed from point (x, y) we see that those points are: $A : (x + dx/2, y)$, $B : (x + dx, y + dy/2)$, $C : (x + dx/2, y + dy)$ and $D : (x, y + dy/2)$. The vector field at each of these points has to be dot-multiplied by the appropriate dr, which are: $(dx, 0)$, $(0, dy)$, $(-dx, 0)$, $(0, -dy)$. We put all these things together and get:

$$\int_\Gamma v \cdot dr = v_x(A)dx + v_y(B)dy - v_x(C)dx - v_y(D)dy$$

We can group the terms with the same differential:

$$\int_\Gamma v \cdot dr = (v_x(A) - v_x(C))dx + (v_y(B) - v_y(D))dy$$

But, what is the difference between $v_x(A)$ and $v_x(C)$? You can write

$$v_x(C) = v_x(A) + dy\frac{\partial v_x(A)}{\partial y}$$

So, it is just $v_{x,y}(A)dy$. By the same reason, the other difference is just $v_{y,x}(B)dx$. Assuming that the square is so small that, when taking derivatives, points A and B are equivalent, you get

$$\int_\Gamma v \cdot dr = (v_{x,y} - v_{y,x})\, dxdy$$

Which is the z-component of the curl! So, we get an interpretation of it: it gives the line integral on a very small square!

What can this be good for? First of all, it can give us a hint to the meaning of the curl. You can think of it as the amount of "local rotation"

But, what happens with big circuits? First of all, consider what happens when you have a circuit and you split it into two, as in figure 2.

So we have the original circuit, Γ, and then the "children": I and II. If we integrate any vector field on I and II and add up the results, we will find some interesting cancellation: the "dividing line" will give contributions with opposite signs for both (dr in one is $-dr$ for the other!), so finally, we get:

$$\int_\Gamma v \cdot dr = \int_{I} v \cdot dr + \int_{II} v \cdot dr$$

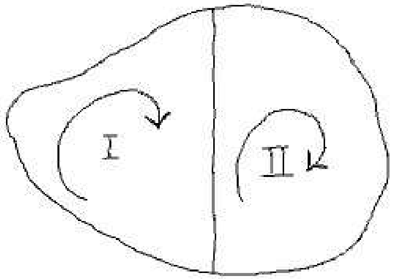

FIGURE 2. A circuit split into two.

The same happens if you split a circuit into many internal parts, as it appears in figure 3. If you sum up the line integral of a vector field on all such small circuits, the internal boundaries cancel out, and you only get the integral on the exterior circuit.

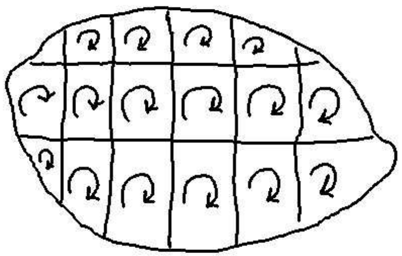

FIGURE 3. A circuit split into many small circuits.

But the internal circuits can be as small as we want. We can make them even infinitesimal! Then, the circulation on each of them is just given by the curl. We get the beautiful result which is known as Green's theorem [1]:

$$\int_\Gamma v \cdot dr = \int_S \nabla \times v \cdot dS$$

where S is any surface bounded by the closed circuit Γ.

E2. Let's check it out. If Γ is the unit square, compute $\int_\Gamma (5 - xy - y^2, 2xy - x^2) \cdot dr$ both directly and using Green's theorem.

[1] Sometimes also Kelvin-Stokes theorem, depending on the dimension of space...

The theorem is more nicely rewritten using a new symbol. If S is any surface, we'll say the curve ∂S is its boundary. Of course, there are two senses of orientation to a curve, which one to choose? We pick up this criterion: the right hand rule. If you run with your fingers along the curve, your thumb points in the normal direction of the surface. With this in mind,

$$\int_S \nabla \times v \cdot dS = \int_{\partial S} v \cdot dS$$

This is our first example of the renowned *Stokes theorem*.

WHEN A VECTOR FIELD IS A CURL?

Let us consider the curl of any vector field, $\nabla \times v$. You know that it is again a vector because of the lucky coincidence that we're in dimension 3. But let's go on, you'll have time to ponder about that in your later inquires in maths... There are some nice properties to it. The first of them comes from Green's theorem. Suppose that we prepare a surface without boundary. What is then the flux of $\nabla \times v$ through it? It's not hard to find out:

$$\int_S \nabla \times v \cdot dS = \int_{\partial S} v \cdot dS = 0$$

because $\partial S = 0$. So: *the flux of any curl field on any closed surface is zero*. In other words: all field lines are closed!

This property of no-flux is nice from a physical point of view: magnetic fields lines are closed, and also for velocity fields of incompressible fluids without sources or sinks. So, both B and v, in such cases, can be written as the curl of *other* vector fields, which are usually called *vector potentials*.

Anyway, let us do the same as before, and try to find out when a given vector field v is the curl of another. We'll work with a very small cube, as in figure 4, with faces parallel to the coordinate planes, and dimension $dxdydz$.

So, the flux of v through that elementary cube is found that way. Let us consider first the two faces which are perpendicular to the x-axis, call them $X+$ and $X-$. We will assume that the faces are so small that the field takes a constant value on each of them. The surface vector dS is pointing in opposite directions: in one $dS_{X+} = (dydz, 0, 0)$ and in the other is $dS_{X-} = (-dydz, 0, 0)$. The flux on these two faces are:

$$\Phi_x = v(X+) \cdot dS_{X+} + v(X-) \cdot dS_{X-} = v_x(X+)dydz - v_x(X-)dydz$$

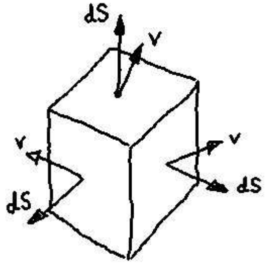

FIGURE 4. Flux through an elementary cube.

OK? But now, the values of v_x at $X+$ and $X-$ are not so different, come on. They're separated a distance dx, so we can do a Taylor development for $v_x(X+)$ in terms of the other:

$$v_x(X+) = v_x(X-) + \frac{\partial v_x}{\partial x} dx$$

So we have, for these two faces:

$$\Phi_x = \frac{\partial v_x}{\partial x} dxdydz$$

Now, for the other two coordinates the same reasoning holds, and we get

$$\Phi = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) dxdydz$$

If the field v is the curl of somebody else, then that scalar quantity in parenthesis should vanish. We will call it **divergence** of a vector field and shorthand it like $\nabla \cdot v$ for obvious reasons. This way:

- If v is the curl of some vector field, then $\nabla \cdot v = 0$.

But from this reasoning we have learnt something that applies to all vector fields, not just "curls". Suppose that you have a large surface, and you split it into small "cubes". By a reasoning analogous to the one above, the sum of the fluxes through all small cubes equals the flux through the exterior surface, because all the inner faces cancel out (for one "cube" it is positive, for the neighbour it is negative). This is true also if the cubes are infinitesimal. This way we get what is known as **Gauss thorem**: if V is any volume and ∂V is its boundary (a closed surface), we get

$$\int_{\partial V} v \cdot dS = \int_V \nabla \cdot v \, dV$$

So, what is the physical meaning of the divergence? We said that a vector is a curl when it has no sources or sinks. And a curl vector field has no divergence. So the divergence must amount for those sources and sinks. Imagine a *compressible* fluid: the velocity field has positive divergence where the fluid is getting expanded, and negative where it is getting compressed. Also, the electric field has positive divergence in places where there is a positive charge.

E3. Consider the unit cube in space, and the velocity field $v(x, y, z) = (x, 0, 0)$. Check Gauss' theorem.

A glimpse of Stokes theorem

(This section is missing at this stage...)