Line and Surface Integration

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Introduction

So far we have done integrations on “full” regions. With that I mean that, if we were working in $\mathbf{R}^3$, the integral was done on a full volume. But there are cases in which we want to integrate a function on a surface, or a line, while being in $\mathbf{R}^3$. For example, when we want to find the area of a sphere, instead of its volume. What can we do?

Length of a Line

Remember one of our main points of logic: it does not matter if we can't carry out the final integral. The most important thing is to reduce the full geometrical or physical problem to such integrals. Computers, approximate methods, etc. will take care of that.

So, let us consider a curve to start with. They can give it to us in explicit –$y = f(x)$– or parametric terms –$r(t) = (x(t), y(t))$–, where t is just an arbitrary parameter. In both cases, let us consider a certain point (x, y) which belongs to the curve, and a small movement (dx, dy) which moves us to another neighbour point on the curve also. So, we have a small *line element*. Its length will be given by Pythagoras theorem:

$$ds = \sqrt{dx^2 + dy^2}$$

If the equation is given in explicit terms, then we can do the trick:

$$dy^2 = dx^2 \left(\frac{dy}{dx}\right)^2$$

So we get

$$ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Hey, that can help! Now, the line element only depends on dx. If the limits of integration are clear, we're done.

An example. Let us compute the length of C, an arc of parabola $y = x^2$, between $x = 0$ and $x = 1$. We have:

$$L = \int_C \sqrt{dx^2 + dy^2} = \int_0^1 dx\sqrt{1 + (2x)^2}$$

(this integral *can* be done analytically, but the result is not specially illuminating.)

If the curve is given in parametric terms, the reasoning is still similar. Let us consider the helix given by $r(t) = (\cos(t), \sin(t), t)$, from $t = 0$ to $t = 2\pi$. In this case the integral will be done analytically. We consider again a point in the curve, (x, y, z), and a small movement (dx, dy, dz), which keeps us in the curve. Then, we have:

$$dx = \frac{dx}{dt}dt = -\sin(t)dt$$

In the same way, $dy = \cos(t)dt$ and $dz = dt$. So, the length of the line element is

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\sin^2(t)dt^2 + \cos^2(t)dt^2 + dt^2} = \sqrt{2}dt$$

So, the total length of the curve in the given interval is

$$L = \int_0^{2\pi} dt\sqrt{2} = 2^{3/2}\pi$$

Note. For sure you have heard about lines which surround a finite area and, yet, their length diverges... they're *fractals*.

Mass of a Wire

Imagine that we are given a wire whose (linear) density is a not homogeneous. Let us consider a parabolic piece of wire, drawing a parabola $y = x^2$ from $x = 0$ to $x = 1$. And let us say that the density depends linearly on x: $\rho(x, y) = kx$. Then, we're asked for the total mass of the wire.

This calculation is very similar to the previous one, in which we had to compute the total length. Again, we're asked to get the length, but each line element is now "weighted" differently: each has density kx. So, we get

$$M = \int_C \rho(x,y)\sqrt{dx^2 + dy^2}$$

So we just insert our density in the previous integral:

$$M = \int_0^1 kx\, dx \left(1 + (2x)^2\right) = \frac{k}{12}(5\sqrt{5} - 1)$$

What we've done is to integrate a scalar function along a line. More interesting and difficult is to integrate *vector* quantities...

Work: Vector Line Integrals

You know that the work exerted by a force F on a body which undergoes a displacement Δr is given by $W = F \cdot \delta r$, with a dot product (i.e.: the result depends on the angle between them!). In physics, the utility of the work is that it gives the change in the kinetic energy of the body. But we won't care much for this now. We'll learn how to compute works.

Let us consider a force field given by $F(x,y) = (0,-y)$, i.e.: it always takes towards the x-axis. We know that a particle has gone, through a straight line, from $(0,0)$ to $(1,1)$. We're asked for the work done by the force on the particle (and, therefore, the change in its kinetic energy). The problem is that the force is changing along the trajectory! So, which force should we take? At the beginning, it is $(0,0)$, and at the end, it is $(0,-1)$.

We should do an integral. We can write it like this:

$$W = \int_C F(x,y) \cdot (dx, dy) = \int_C F_x(x,y)dx + F_y(x,y)dy$$

where C is again the trajectory. Seems to be easy, but something should be made clear: dx and dy are not unrelated! That's because they refer to the small displacement done by the particle while moving along the curve C. So, in our case, C is the line $y = x$, so $dy = dx$. About the integration limits, let us express everything in terms of x, and say it is going to be from $x = 0$ to $x = 1$:

$$W = \int_0^1 dx\,(0 \cdot dx - x \cdot dx) = -\frac{1}{2}$$

The negative sign, in physical terms, means that the particle loses kinetic energy in the process, i.e.: it is decelerated by the force.

OK, let's see a more complicated one. Let us consider the force field $F(x,y) = (-x,-y)$, which always points towards the origin, and the particle going in a trajectory which is a parabola arc, $y = x^2$, from $x = 0$ to $x = 1$. Then, our general formula gives

$$W = \int_C F(x,y) \cdot (dx, dy) = \int_C (-x dx - y dy)$$

Again, dx and dy are not independent, since they are a movement along the parabola. So, we say, since $y = x^2$, that $dy = 2x dx$, and the integration limits are those for x:

$$W = \int_0^1 dx\left(-x\ dx - x^2\ 2x\ dx\right) = -1$$

As we said before, a curve can be parametrized in any way we may desire. For example, imagine that, with the same field, you have to find the work done on a particle going from $(1,0)$ to $(-1,0)$ through a circumference arc with center $(0,0)$. The half-circumference can be parametrized in many ways. The most straightforward may be to say that $y = \sqrt{1-x^2}$. But a more clever one may be to do it like this: $r(t) = (\cos(t), \sin(t))$, with $t \in [0,\pi]$. So, $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Then, $dx = -\sin(t)dt$ and $dy = \cos(t)dt$. We get

$$W = \int_C F(x,y) \cdot (dx, dy) = \int_0^{\pi} dt(-\cos(t), -\sin(t)) \cdot (-\sin(t)dt, \cos(t)dt)$$

This integral is zero, which you can easily check in the formula. The real reason is that the force is always perpendicular to the displacement.

Introducing Differential Forms

A differential form is anything which can go under an integral sign. An object of the form $(x^2 + y)dxdydz$ is a *volume form*, or 3-form, and can go under an integral sign specified for a region in $\mathbf{R}^3$. In order to imagine it, you have to split mentally the space in infinitesimal cubes of volume $dxdydz$, and inside each of them assign a value, which is $x^2 + y$. Then, to integrate on a region is just to sum those values for the small cubes which fit inside.

The objects under the integrals along lines which we have just learnt to do are called 1-forms. For example, $y^2\,dx - xy\,dy + 2dz$. It has to be integrated on a line, in order to give a number. Imagining them is a little bit more difficult. Picture yourself a very tight cubic wireframe, with lines which are parallel to the axes, x, y and z, cutting each other at regular spaces. It splits the space into the small cubes we discussed before. Each cube has three types of edges: those parallel to the x-axis, with length dx, etc. Now, assign to each infinitesimal edge along the x-axis the value y^2 at its position, the value xy to each y-axis edge and the value 2 to each z-axis edge.

Now, how to integrate on a line? I told you the wireframe was really tight. So tight that each small cube has infinitesimal size. Now, you have to "approximate" the trajectory using these edges.

Of course, this is not the full story, just an introduction to it. For one thing, you have seen 3-forms and 1-forms... but for sure you're missing the intermediate step, the 2-forms. Obviously, 2-forms are integrated on surfaces, which constitute the rest of this text...

Parametrizing a Surface

A surface is a much more complicated object than a line. For one thing, surfaces can have "real curvature". With this I mean the following: A piece of thread can always be straightened out without tearing it apart. But try this with a half-sphere. You can't lay it flat without tearing it. We already said that a surface can always be parametrized with two variables, which are usually called u and v:

$$r(u,v) = (x(u,v), y(u,v), z(u,v))$$

This describes a surface. We start with this example

$$r(u,v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$$

With $u \in [0,\pi]$ and $v \in [0,2\pi]$. This formula has a terrible look until we remember the conversion from spherical to cartesian coordinates! Then, if $u = \theta$ and $v = \phi$, we see that we're describing an object with radius $r = 1$ and the angles covering all possibilities. Obviously, this is just a sphere.

Another nice example is given by

$$r(u,v) = (u\cos(v), u\sin(v), v)$$

With $u \in [0,1]$ and $v \in [0,\infty)$. In order to see this, we fix a value for u, say $u = 1$. Then we have a curve,

$$r(v) = (\cos(v), \sin(v), v)$$

This curve is a *helix* with radius 1. So, by changing u we change the radius and we get many helices. The figure is called a *helicoid*, and is drawn in the next figure.

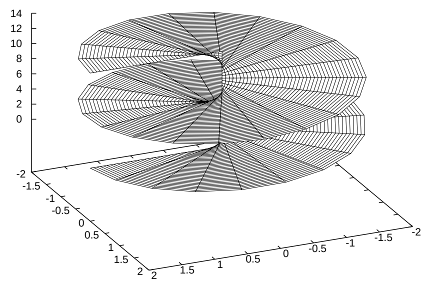

Figure 1. A helicoid.

Don't worry, we just show this figure because it's nice, but we're not going to integrate on such complex surfaces! Ah, and, obviously, a fine parametrization is given when we have $z(x,y)$, because then it's as easy as this:

$$r(u,v) = (u, v, z(u,v))$$

But, also obviously, not all surfaces can be cast in the form $z(x,y)$. The helicoid is a nice example, do you see why?

Area of a Surface

Now, let us develop the mathematical tools so find the area of a surface. We can decompose the surface in small squares of size $dudv$. Then, we can find the area of each $dudv$ square and add them together.

If we fix $u = \text{const}$, then we get a curve parametrized by v. For example, in

$$r(u,v) = (u\cos(v), u\sin(v), v)$$

if $u = u_0$, we get $r(v) = (u_0\cos(v), u_0\sin(v), v)$, which is a helix. If we fix $v = \text{const}$, then we get another curve: $r(u) = (u\cos(v_0), u\sin(v_0), v_0)$. These are straight lines parallel to the xy-plane coming out of the z-axis with angle v_0.

So, if $u = \text{const}$, a movement of magnitude dv along the corresponding curve gives rise to the *real space* movement:

$$dr = (dx, dy, dz) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) dv$$

We get a "vector field", tangent to the v-curves, which we will call T_v. (OK, really that's a 1-form.) The same way, we write down the vectors which are tangent to the u-curves. Writing everything together:

$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \qquad T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

You can think that T_u and T_v are the *sides* of the small squares $dudv$ in which we divide the surface. Think of vector $T_u \times T_v$, i.e., the cross product of them both. As a vector, it points in the *normal* direction to the surface. The reason is that both T_u and T_v are pointing *along* the surface. But its norm is more interesting: $\|T_u \times T_v\| dudv$ is the *area* of the small square $dudv$.

This allows us to compute areas in a straightforward way. If S is the surface, with integration limits given in u and v, then

$$A = \int_S dudv \, \|T_u \times T_v\|$$

Very often, we will write $dS = \|T_u \times T_v\| dudv$.

OK, an example is in order. Let's compute the area of the spherical surface. We already have a parametrization:

$$r(u, v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$$

So we get the two tangent vectors:

$$\begin{cases} T_u = (\cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)) \\ T_v = (-\sin(u)\sin(v), \sin(u)\cos(v), 0) \end{cases}$$

Now we get the normal vector

$$T_u \times T_v = \left(\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u\right)$$

The modulus squared, after some massaging (yes, it's slow and boring, but it's straightforward) is simply $\|T_u \times T_v\| = \sin u$. This allows us to do the integral:

$$A = \int_0^{\pi} du \int_0^{2\pi} dv \sin u = 4\pi$$

This is the great result for the area of a sphere of radius one. As it is logical, if the radius is r, a simple scaling argument gives the known $4\pi r^2$.

Flux: Vector Integrals on Surfaces

Consider a surface S immerse in a fluid in motion. At each point in space, the velocity of the fluid is given by a vector field, $v(x, y, z)$. This velocity need not be the same in all points. The flow could have vortices and do all kinds of strange things. Now we're asked how much fluid crosses the surface per unit time, also called *flux*. How to find it?

Let's consider, to start, a uniform velocity field, $v(x, y, z) = (v_x, v_y, v_z)$, and a rectangle of area $|S|$, whose orientation is given by a unitary normal vector N. It is customary to summarize all that information in a single vector S, whose direction is N and whose norm is $|S|$. We will do so. Then, the flux through the rectangle is given by $v \cdot S$. No physical proof is given, but a nice argument follows: of course, the flux must be proportional both to the magnitude of v and S, and also should vary with the angle between them. As a matter of fact, it should go like the velocity into the "projected area", into the direction of v.

So, let us now consider a general surface and a general velocity field. The flux is now given by the following integral:

$$\Phi = \int v \cdot dS$$

where dS is the small vector, whose modulus is the area of the small surface element, and whose direction is normal to it. Of course, a certain orientation of the surface is assumed (there are two normals!).

So, let's go with an example. Let us consider a square on the xy plane, made from points $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, with the normal pointing upwards. The surface can be parametrized this way:

$$r(u, v) = (u, v, 0)$$

with $u \in [0, 1]$, and $v \in [0, 1]$. The normal vector is trivial: $N = (0, 0, 1)$, and the surface element is $(0, 0, 1)dudv$. Now we ask about the velocity field, of course. Let us say that it is given by

$$v(x, y, z) = (x^2, 2xy, y + z)$$

So the total flux through the surface is:

$$\Phi = \int_S v \cdot dS = \int_0^1 du \int_0^1 dv (x^2, 2xy, y + z) \cdot (0, 0, 1)$$

Remember that $x = u$, $y = v$ and $z = 0$, so

$$\Phi = \int_0^1 du \int_0^1 dv v = 1/2$$

In physics, fluxes appear also in electromagnetism, since two of Maxwell's laws are expressed very neatly in terms of the flux of the electric and magnetic fields through closed surfaces. Let us consider another example of flux.

Suppose that you have a static field given by $v = (y^2, xy, xz)$ and you're asked to find its flux through the piece of surface given by the paraboloid $z = x^2 + y^2$, with $x \in [0, 1]$ and $y \in [0, 1]$. First, we will parametrize the surface this way:

$$r(u, v) = (u, v, u^2 + v^2)$$

with $u \in [0, 1]$, $v \in [0, 1]$. Now we find the tangent vectors:

$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (1, 0, 2u)$$

In the same way, $T_v = (0, 1, 2v)$. The normal vector can be found by taking the cross product:

$$dS = \|T_u \times T_v\| du dv = (-2u, -2v, 1) du dv$$

Notice that we have to choose the global orientation of the surface, since $dS = (2u, 2v, -1)$ is also a good choice. Let us stick to the previous one, and say that the normal points "upwards". Then, the integral is done:

$$\Phi = \int_S v \cdot dS = \int_0^1 du \int_0^1 dv (y^2, xy, xz) \cdot (-2u, -2v, 1)$$

of course, we have to substitute x, y and z in terms of the parametrization:

$$\begin{aligned} \Phi &= \int_0^1 du \int_0^1 dv (v^2, uv, u(u^2 + v^2)) \cdot (-2u, -2v, 1) \\ &= \int_0^1 du \int_0^1 dv (-2uv^2 - 2uv^2 + u^3 + uv^2) = -1/2 \end{aligned}$$

E1. Find the flux of a constant vector field pointing on the z-axis through the upper half-sphere of radius one. Is the result natural?