Taylor Series

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Estimated Time of Arrival

You're downloading a file from... somewhere. You see the speed in your screen, and you want to guess when it's going to finish, the *estimated time of arrival* (ETA). Let's say that you must download A Mb, and the speed is v Mb/s. Then, your ETA is just A/v, right?

But life is not so simple. You realize that the speed is decreasing. It is "accelerating" at a rate a Mb/s every second (Mb/s^2). So you should "update" your estimate. How?

Let us call $f(t)$ the function which gives us the amount of information downloaded as a function of time. Then, our first approximation was to consider a linear $f(t) = A - vt$. In the second case, we know the curve should "bend" somehow.

We'd like to see a function $f(t)$ which has a constant "acceleration". But the acceleration is the *derivative* of the velocity, so the derivative of the derivative, so the *second derivative* of the function. We want $f''(t) = c$. Which functions, when twice derivated yield a constant? The parabolas.

So we propose $f(t) = A - vt - kt^2$, for some k. A little work (do it!) shows that $k = a/2$. So, $f(t) = A - vt - at^2/2$. This way you update your estimate for the ETA by solving $f(t) = 0$.

But real life is *always* more complicated. Now you find that the acceleration is also changing with time! This is real life. How to give an accurate value for the ETA? This is the problem that we're trying to solve.

Approximating

When we introduced the derivatives, we showed how the tangent line can be used as a local approximation to the real function. By local we mean, near the "tangent point". But there is a problem: the real function "bends", it is curved, while the tangent line is straight, so they detach more and more as you go away from that tangent point.

We can do better. What about trying to use, instead of the straight line,

another function which is also simple but it also *bends*? I don't know about you but, for me, the second best after the straight line, is the parabola.

So, let's review the idea of the tangent line, and we'll see how to generalize it. Consider the original function $f(x)$, a point $x = x_0$ and the tangent line $p_1(x) = a_0 + a_1(x - x_0)$ fulfills two conditions:

(c_1) At $x = x_0$, both functions take the same value: $p_1(x_0) = a_0 = f(x_0)$.

(c_2) At $x = x_0$, both functions have the same *slope*: $p_1'(x_0) = a_1 = f'(x_0)$.

This way we find the explicit expression:

$$p_1(x_0) = f(x_0) + f'(x_0)(x - x_0)$$

We could say, OK, let's add one more level, say $p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$, and ask *three* conditions. The two previous ones and a third one.

(c_3) At $x = x_0$, both functions have the same *acceleration*.

The two first conditions give the same result! That's nice. And the third yields:

$$p_2''(x_0) = 2a_2 = f''(x_0)$$

Everything is nice, but the factor two... Is there any easy way to explain it?

The Derivatives of a Polynomial

Now imagine a polynomial of this form:

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

At $x = x_0$, $p_n(x_0) = a_0$, that's easy. Now, the derivatives:

$$p_n'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

We evaluate at $x = x_0$ and all the terms vanish except the first one, so $p_n'(x_0) = a_1$. Now, the second derivative:

$$p_n''(x) = 2a_2 + 3 \cdot 2a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2}$$

Evaluating at $x = x_0$ we get $p_n''(x_0) = 2a_2$. For the third derivative, we get

$$p_n'''(x) = 3 \cdot 2a_3 + \cdots + n(n-1)(n-2)a_n(x-x_0)^{n-3}$$

So, $p_n'''(x_0) = 3 \cdot 2a_3$. It's easy to see that, in general,

$$p_n^{(k)} = k!a_k$$

Taylor Polynomials

Now we have all the parts of the puzzle, we only have to put them together. We know that, for a polynomial of the previous form, the k-th derivative is $k!a_k$. Given a function $f(x)$, we will design a polynomial whose first n derivatives coincide at x_0. The more derivatives we fit, the more the two functions will look alike. The explicit formula for Taylor's polynomial is this:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Remember that the numbers $f(x_0)$, $f'(x_0)$ are only numbers, not functions!! We can summarize that formula in a single expression:

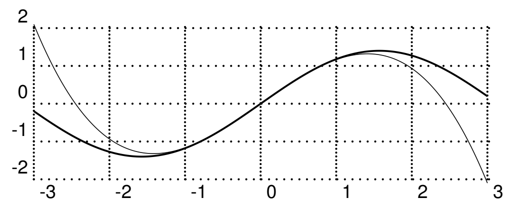
$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Buff... Let's do an example, otherwise it'll never be clear. We take the $f(x) = \sin(x)$ function. We'll take $x_0 = 0$ for simplicity. Now, let's evaluate the function and its derivatives.

$$\begin{aligned} f(x) &= \sin(x) & \to \quad & f(0) = 0 \\ f'(x) &= \cos(x) & \to \quad & f'(0) = 1 \\ f''(x) &= -\sin(x) & \to \quad & f''(0) = 0 \\ f'''(x) &= -\cos(x) & \to \quad & f'''(0) = 1 \end{aligned}$$

Enough for now. Let us make up the polynomial, with those data. Using a direct application of the previous formula we get:

$$\sin(x) \approx x - \frac{x^3}{6}$$



Do you see it? Now, let's plot the two functions so that you see the differences:

See? They really look alike near the zero!

If we fit more and more derivatives, the polynomial and the function look more and more alike. Thus, we might ask what happens if we write a polynomial with "infinite order". How to do it? Doing the infinite derivatives.

A little thought shows that the derivatives of $f(x) = \sin(x)$ at $x = 0$ follow a certain pattern: $0, 1, 0, -1$ and then repeating. So, we may write the whole polynomial, for $n \to \infty$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

If we want to write this in a closed formula, we can:

$$\sin(x) = \sum_{i=1}^{\infty} (-1)^i \frac{x^{2i-1}}{(2i-1)!}$$

This is not called a *polynomial*, since polynomials should always have finite order. This is called a *Taylor series*.

Analytical Functions

Although this last formula is not going to be much used. But the funny thing is that, if all derivatives coincide, that "infinite polynomial" (we prefer to call it series) should *coincide* exactly with the sine function, right? OK, we have to be careful with this. It's not true for all functions, but it's true for many. When it does, when the function coincides exactly with its (infinite order) Taylor polynomial, we call it *analytical*.

What does it mean to be analytical? A lot! Consider it this way: if a function is known to be analytical, it can be rebuilt from its value at a point and all its derivatives *at that very point*. So, in a sense, a local study, only taking into account the vecinity of that point, allows us to rebuild the function.

It's as if you could rebuild the image of a person from a skin cell! It means, basically, the function is *amazingly regular*, and it bears no surprises.

For example, if a function has a jump ($f(x) = 0$ if $x < 0$ and 1 if $x \geq 0$), then the function is not analytical. If you study its value and all its derivatives, say at $x = -1$, then you will think that it is function zero... and it is not. It was not possible to predict the "jump" in the function from a study, no matter how careful, of the vicinity of $x = -1$.

On the other hand, let's take $f(x) = (x-1)^{-1}$. It is analytical everywhere except $x = 1$. This means that, by a careful study of the function and all its derivatives at $x = 0$ you can guess that it has a pole at $x = 1$. This is one of the magical things of higher mathematics...

Computing Taylor Series

There is a basic trick in order to compute Taylor series: use whatever you have at hand and any way you reach is valid. OK, this is not *a trick*, is more like a philosophy... Let's try to put it to practice.

Let's start with the simplest possible series: $1 + x + x^2 + x^3 + \cdots$. We can sum that, it's just a geometric series. If $|x| < 1$,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

So we get the first conclusion, the Taylor polynomial of $(1-x)^{-1}$ is just $\sum_i x^i$. OK. Now, we can derivate the polynomial, and that would be the same as derivating the function, no?

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

So, you see what I mean. Any way is valid. You can follow the normal derivation path, and you'll find exactly the same.

But we can also take the *anti*-derivative of $(1-x)^{-1}$ (OK, integral, that's the same), and get $-\log(1-x)$ (plus a possible constant). Doing also the *anti*-derivative for all terms of the polynomial we get:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x) + K$$

Substituting both terms at $x = 0$ we get that $K = 0$. So, another polynomial, almost for free!

OK, let's do some derivatives, but without much effort. The function which, under derivation, remains always the same $\exp(x)$. All the derivatives at $x = 0$ are equal to 1, so we have

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(where we've used the usual convention that $0! = 1$.) This way we get another way of computing e:

$$e = \exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Now, let's go for a polynomial evaluated away from zero. Let us consider, for example, $f(x) = x^3 + 2x^2 + 5$, and we need to express that same polynomial as a sum of powers of $(x-2)$. How to do it? Easy: we take the Taylor series at $x = 2$. So we evaluate the derivatives there:

$$\begin{aligned}
f(x) &= x^3 + 2x^2 + 5 &\quad\to\quad& f(2) = 23 \\
f'(x) &= 3x^2 + 4x &\quad\to\quad& f'(2) = 20 \\
f''(x) &= 6x + 4 &\quad\to\quad& f''(2) = 16 \\
f'''(x) &= 6 &\quad\to\quad& f'''(2) = 6
\end{aligned}$$

From that moment on, all the derivatives are *zero* so the Taylor series *terminates*. We can write it in full:

$$f(x) = 23 + 20(x-2) + \frac{16}{2}(x-2)^2 + \frac{6}{3!}(x-3)^3$$

If a Taylor series is computed around $x = 0$ it is customary to give a different name to it, and call it a *Maclaurin series*.

Now, a different story. What about using Taylor polynomials to compute square roots? OK, we have a try by normal derivation of $f(x) = x^{1/2}$. But the first derivative gives a surprise: $f'(x) = (1/2)x^{-1/2}$, so $f'(0)$ is... infinite? Yes, it is. The parabola x^2 has a minimum at zero, so its inverse function has infinite slope at the origin!

Therefore, we have to do something. What about developing around $x = 1$? We also know how to compute everything there... Please, do the calculations yourself to get convinced that, around $x = 1$,

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \cdots$$

First, let us estimate $\sqrt{2}$ from there, using $x - 1 = 1$, of course. We get $1'398$, OK, not too bad. It improves with more terms, of course.

Can we get an estimate of the error when approximating a function with its Taylor polynomials? Yes, but we'll need integral theory for that, so we leave it until then...

If you have a product of functions, then you should multiply the series for each of them. For example, to find the polynomial, up to order 3, of $f(x) = e^x \sin(x)$, we do not derivate. We write the series for them both:

$$f(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right)$$

We do the products, only retaining powers equal or less than three:

$$f(x) = x + x^2 + \frac{x^3}{3} + \cdots$$

Obviously, we have not generated the "full pattern" (i.e.: the series), but we did what we were asked for.

Another example would be to compute the series of a composite function. For example, $f(x) = \exp(\sin(x))$. It is quite different. In this case, we substitute the series for the sine *inside* each x in the series for $\exp(x)$:

$$f(x) = 1 + \sin(x) + \frac{\sin(x)^2}{2} + \frac{\sin(x)^3}{6} + \cdots$$

So, now we substitute, some algebra, and up to degree 3 we get:

$$f(x) \approx 1 + x + \frac{x^2}{2} + \cdots$$

(and the term in x^3 vanishes).

Limits using Taylor

Consider the following limit:

$$L = \lim_{x\to 0} (1 + x - \sin(x))^{1/x^3}$$

It has the form 1^∞, therefore it will yield, most likely, e^α for some α... But near the vicinity of $x = 0$, $\sin(x) \approx x - x^3/6$, so we have

$$L = \lim_{x\to 0} \left(1 + x - x + \frac{x^3}{6} + \cdots\right)^{1/x^3}$$

Wonderful, the limit is now trivial: $L = e^{1/6}$. Using L'Hôpital and other methods, the problem becomes much harder...

So, the lesson: try to approximate a function by its Taylor series, and find out the leading term. *Be careful*: don't neglect terms unless you are *sure* they're useless! For example, if we had substituted $\sin(x) \approx x$ naively, the limit would have been wrong!

How Much Error?

So, we know we can approximate $f(x)$ by its Taylor polynomial of order n. But, how good is the approximation? Here we won't give a proof for this, but clever application of mean value theorem, or a less clever but faster application of integration by parts gives you an estimate for the error made. The difference between the n-th order Taylor polynomial around $x = a$ and the real function is equal to the *remainder*:

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$$

for some value of t in the interval $[a, x]$. So, all you have to do to estimate the error is to give a bound for the $n+1$-th derivative of the function.

An example. We approximate $\sin(1)$ by the 5-th order polynomial:

$$\sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} = 0'84167$$

In order to evaluate the error we compute the remainder. The sixth derivative of the sine is the cosine again, so

$$R_{5,0} = \frac{\cos(t)}{6!}x^6$$

But $x^6 = 1$ in our case. We have to bound the cosine function in the $[0, 1]$ interval, which is easy: $\cos(t) \leq 1$ for sure. So we get $R_{5,0} < 1/6! = 0'00139$. Effectively, the real value for $\sin(1) \approx 0'84147$, and the real difference is $0'0002$.