Functions in Space

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We're tired of functions going from $\mathbf{R}$ to $\mathbf{R}$. We want action! We want functions which are able to describe shapes of 3D objects, which depend on more than one parameter, twisted curves and surfaces. How can we go?

In general, we can define functions going from $\mathbf{R}^n$ to $\mathbf{R}^m$, for whichever values of n and m. This means that a function f may take, for example, three inputs and return two outputs. This f will be said to go $f : \mathbf{R}^3 \mapsto \mathbf{R}^2$. With functions $\mathbf{R} \mapsto \mathbf{R}$, we know how to get a graphical interpretation, but what about these other functions? Today, we'll try to show how to view a few of them...

A useful advice: please, try to visualize the functions without a computer in your first try. But, after you get some mastery, it is a very nice idea to get some visualization software. It is worth to try to learn how to use them. I myself like `gnuplot` a lot, and it is free. Visit `http://www.gnuplot.info`.

Curves

A function $f : \mathbf{R} \mapsto \mathbf{R}^n$, often with $n = 2$ or $n = 3$, is said to be a curve. The input value is a single parameter, which, which we shall call t. You can think of it as time, and the full curve to be the trajectory of a moving particle. But remember that this is not the only possible interpretation!

So, let's start with a simple one: $f : \mathbf{R} \mapsto \mathbf{R}^2, f(t) = (r\cos t, r\sin t)$, for some fixed value of r. I guess you might recognize it, but let us assume you don't. What to do first? Let's try a few points: $f(0) = (r, 0)$, $f(\pi/2) = (0, r)$, $f(\pi) = (-r, 0)$, $f(3\pi/2) = (0, -r)$, $f(2\pi) = (r, 0)$... and then it repeats. If we're clever, a simple check may solve all our doubts about the function. For any time t, what is the distance of the point (x, y) to the center? $x^2 + y^2 = r^2(\cos^2 t + \sin^2 t) = r^2$. Always the same? Hey! That's just a circle!!

E1. Try to plot $f : \mathbf{R} \mapsto \mathbf{R}^2, f(t) = (t\cos t, t\sin t)$.

E2. This circumference is traversed counterclockwise. Can you give the equation of a clockwise traversed circle?

Now, for more action, let's move on to a 3D example. $f : \mathbf{R} \mapsto \mathbf{R}^3, f(t) = (\cos t, \sin t, t)$. What can it be? The first two components look like we're moving on circles, but the z-component is... advancing, right? So, it must be a *helix*, the outline of a spiral staircase. You can see it in figure 1.

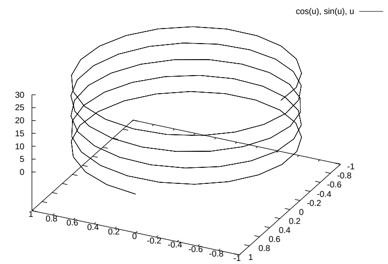

FIGURE 1. Curve given by $f(t) = (\cos t, \sin t, t)$, a helix.

E3. Give the equation of the mirror image of the helix in figure 1.

E4. A projectile is launched with speed $(1, 1, 0)$ from the point $(0, 0, 10)$. Give the equation of its trajectory.

E5. What can you say about a curve with equation of the form $(v_x t + b_x, v_y t + b_y, v_z t + b_z)$?

Surfaces

A function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ can be considered to depict a certain surface. Why? This function takes two parameters and returns one. You can think that it takes the coordinates x and y as input and return the z as output. As an example, I'll show you a complicated function in figure 2. Visualizing such a function from the formula is rather difficult task. We'll start with simpler ones!

In order to visualize a function of the form $z(x, y)$ there are many approaches. None is perfect, you should choose depending on your function at hand.

First of all the "vertical cuts". You may consider your $z(x, y)$ as a function of x, with y fixed to different values. One example: Consider $z(x, y) = x^2 + y^2$. For $y = 0$ it is just a parabola, $z = x^2$, right? If you fix $y = 1$, then it is another parabola, but "raised" $y^2 = 1$. If $y = -1$, the same! We realize that y

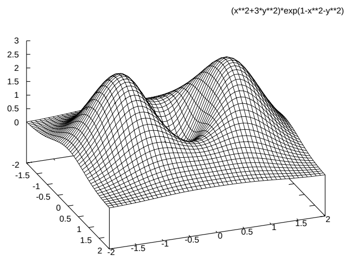

FIGURE 2. Graph of $z(x, y) = (x^2 + 3y^2)\exp(1 - x^2 - y^2)$.

and $-y$ are the same... OK, now for $y = \pm 2$ you get again the same parabola, but raised "more", now 4 units... We can try to show it in figure 3.

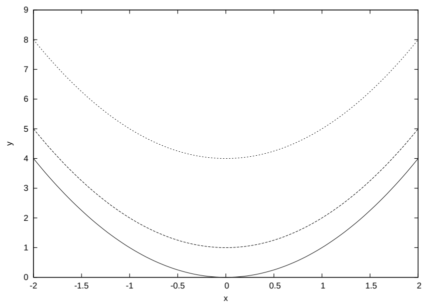

FIGURE 3. Trying to imagine $z(x, y) = x^2 + y^2$ using vertical cuts. The lowest parabola corresponds to $y = 0$. The next one to $y = \pm 1$ and the top one to $y = \pm 4$.

Now imagine yourself going backwards, from $y = 2$ to $y = -2$. Then you see how the parabola goes down and then up. You may imagine something like the *paraboloid* in figure 4.

This paraboloid function has many ways of visualization. Another useful one is the *level sets* (aka "horizontal cuts"). What is that? It's the way you normally visualize a terrain, which is really a $z(x, y)$ surface, right? What you do is to draw the lines $z(x, y) = c$ for some representative values of c. This is also called a *contour plot*. We'll exemplify with the paraboloid. Let's choose $c \in \{0, 1, 4\}$. Then we have the level sets:

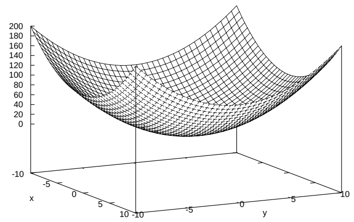

FIGURE 4. The paraboloid of figure 3, in its full splendor.

$$\begin{cases} c = 0 & \to \quad x^2 + y^2 = 0 \quad \text{A single point, } (0, 0) \\ c = 1 & \to \quad x^2 + y^2 = 1 \quad \text{A circumference of radius } 1 \\ c = 4 & \to \quad x^2 + y^2 = 4 \quad \text{A circumference of radius } 2 \end{cases}$$

So, we plot all these on figure 5. From it you can also visualize the paraboloid of figure 4. You just have to mentally "raise" each circle to the appropriate height!

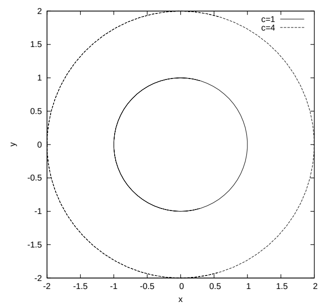

FIGURE 5. Sketchy contour plot of the paraboloid.

E6. Now try the same techniques (vertical cuts and level sets) in order to get an picture of the surface $z(x, y) = x^2 - y^2$.

E7. Same for $z(x, y) = \sin(x) + \sin(y)$. And for $z(x, y) = \sin(x) \cdot \sin(y)$.

And even one more visualization technique! A point on the plane with coordinates (x, y) is at a distance $r \equiv \sqrt{x^2 + y^2}$ from the origin $(0, 0)$, right? So the equation of the paraboloid is just $z(x, y) = r^2$. In other words: the value of the function only depends on the distance to the center. Therefore, it is equal in all points of the same circumference (centered on $(0,0)$)! The resulting surface is known as a *revolution* surface.

So, there is a very nice way to visualize them. Just imagine that u plot a function $z(r) = r^2$. Just a nice parabola. And now, u plot it on the (say) yz “wall”, like in figure 6. But only for positive r, eh?

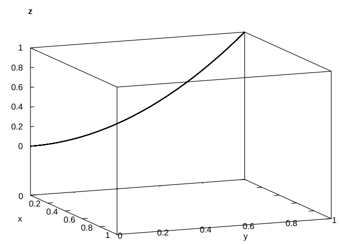

FIGURE 6. Trying to represent the paraboloid as a revolution figure.

Now you rotate the parabola around the z-axis. At least mentally! :) You get something similar to figure 7.

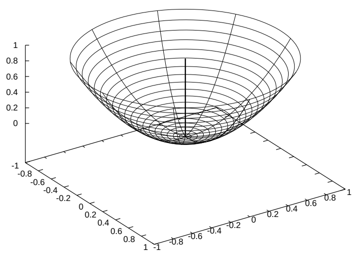

FIGURE 7. Now, really, the paraboloid as a revolution figure.

E8. Now try to represent $z(x,y) = 10 - r$, $z(x,y) = \sin(r)/r$ and $z(x,y) = r^4 - 2r^2$.

E9. Try to give an analytical expression for the function represented in figure 8.

Q1. Do you think that every surface in $\mathbf{R}^3$ can be represented like a function $\mathbf{R}^2 \mapsto \mathbf{R}$? What about a sphere?

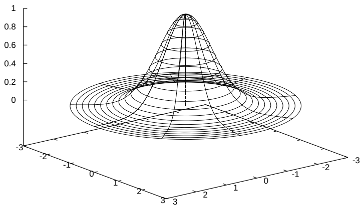

FIGURE 8. Guess, guess.

VECTOR FIELDS

Function $\mathbf{R}^2 \mapsto \mathbf{R}^2$ can be visualized in many ways. You can consider them as mapping points of the plane into other points, so it might be interesting to find out how certain special regions transform. For example, the unit square or the unit circle. As an example, we'll consider the function $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ given by $f_x(x,y) = x$, $f_y(x,y) = x + y$.

Let us consider the unit square, marked by the points $A = (0,0)$, $B = (1,0)$, $C = (1,1)$ and $D = (0,1)$. The images of those points are $A' = (0,0)$. $B' = (1,1)$, $C' = (1,2)$ and $D' = (0,1)$. Let us plot the image of the unit square in figure 9.

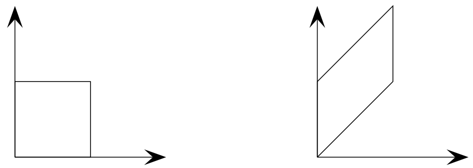

FIGURE 9. Transform of the unit square under the action of function f defined in the text.

E10. Find in a similar way the action of the function $f_x(x,y) = y$, $f_y(x,y) = -x$.

E11. Find the analytic expression of a function $\mathbf{R}^2 \mapsto \mathbf{R}^2$ that expands around the center a factor 2 and rotates, also around the center, $\pi/2$ counterclockwise.

Obviously, such a way of visualizing $\mathbf{R}^2 \mapsto \mathbf{R}^2$ functions is very useful for image processing, as you can guess. But there is another type of application in physics and engineering which is very useful: that of vector fields. I mean: to place a “small arrow” in each point of space. They may stand for forces, electric fields, velocity of a fluid, whatever!

For example, imagine that we’re told that a fluid in a certain region has a velocity at each point given by $v_x(x, y) = -Ky$, $v_y(x, y) = Kx$, for some constant K. How to visualize it?

A nice idea might be to select some relevant points and try to draw the “small arrow” on each of them... Do try it as an exercise, ok?

E12. Pick up the points $(0, 0)$, $(1, 0)$, $(2, 0)$ and its counterparts in the other quadrants, and try to figure out how is the vector field. Does it decay with distance or not?

E13. Design a similar vector field, with the same symmetry, but which decays with distance.

E14. Plot $E_x(x, y) = Kx/r$, $E_y(x, y) = Ky/r$. Give an interpretation in terms of some physical phenomenon.

Of course, we have just scratched the topic. When you get more analytical tools, we’ll review the visualization ideas, of course.